Static Equilibrium Analysis of Multibody Systems

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CERTIFICATE

This is to certify that the work under the thesis titled **Static Equilibrium Analysis of Multibody Systems** by Jaydeep L. Gaikwad has been carried out under my supervision and this work has not been submitted elsewhere for a degree.

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Dedicated to

my parents

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Jaydeep L. Gaikwad

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Abstract

Multibody mechanical systems find a tremendous number of applications in various fields like I. C. engines, automobile transmission, robotics, structural supports etc. Analysis of such systems leads to massive algebraic manipulations in constructing equations of motion. Since these equations are highly nonlinear, the prospect of obtaining closed-form solutions is remote, except in very simple cases. The objective of computer-aided analysis is to formulate and solve such problems using digital computer. Several computer programs for analysis of mechanical systems are developed since 1970's.

In the present work, the static equilibrium analysis of compliant mechanical systems under the action of external loading is attempted. Statically determinate as well as statically indeterminate systems are analysed. Relative coordinates are used as generalised coordinates. Using relative coordinates not only requires less number of generalised coordinates, but it also makes analysis computationally efficient. The system is modeled using homogeneous transformation matrices and loop closure equations. For the solution of the equilibrium problem, potential function (Φ) is used. Minimisation of Φ, subject to rigid constraints gives equilibrium configuration. The approach is tested on some planar and spatial mechanical systems.

Chapter 1

Introduction

A mechanical system is defined as a collection of bodies (or links) in which some or all of the bodies can move relative to one another (Fig. 1.1). Broadly they are classified into two types, *mechanisms* with degrees of freedom greater than zero and *structures*, with zero degree of freedom.

Mechanical systems find wide range of applications in various fields. A few examples are automobile transmission, robotics, structural supports, I. C. engines, suspension systems, aviation etc.

1.1 Simulation of Mechanical Systems

The process which allows the engineer to study the response of an already existing system to a known excitation is called *analysis*. This requires a complete knowledge of the physical characteristics of the mechanical system, such as material composition, shape and arrangement of parts. Due to the nonlinear nature of analysis problems, mechanical designer has traditionally resorted to graphical techniques and physical models for system analysis. As might be expected, such methods are limited in generality and rely heavily on the designer's intuition. With the advent of digital computers, engineers started using the computer and available numerical methods to solve highly nonlinear analysis problems.

Factory automation is one of the major objectives of modern industry. Various parts of the product are designed in the Computer-Aided Engi-

neering branch. Computerised product design requires such capabilities as Computer-aided analysis (CAA), Computer-aided drafting, design sensitivity analysis and optimisation. The Computer-aided analysis capability serves as part of the design process. It allows the engineer to simulate the behaviour of a product and modify its design prior to actual production. Thus Computer-aided analysis can be used as a partial substitute for laboratory or field tests in order to reduce the cost.

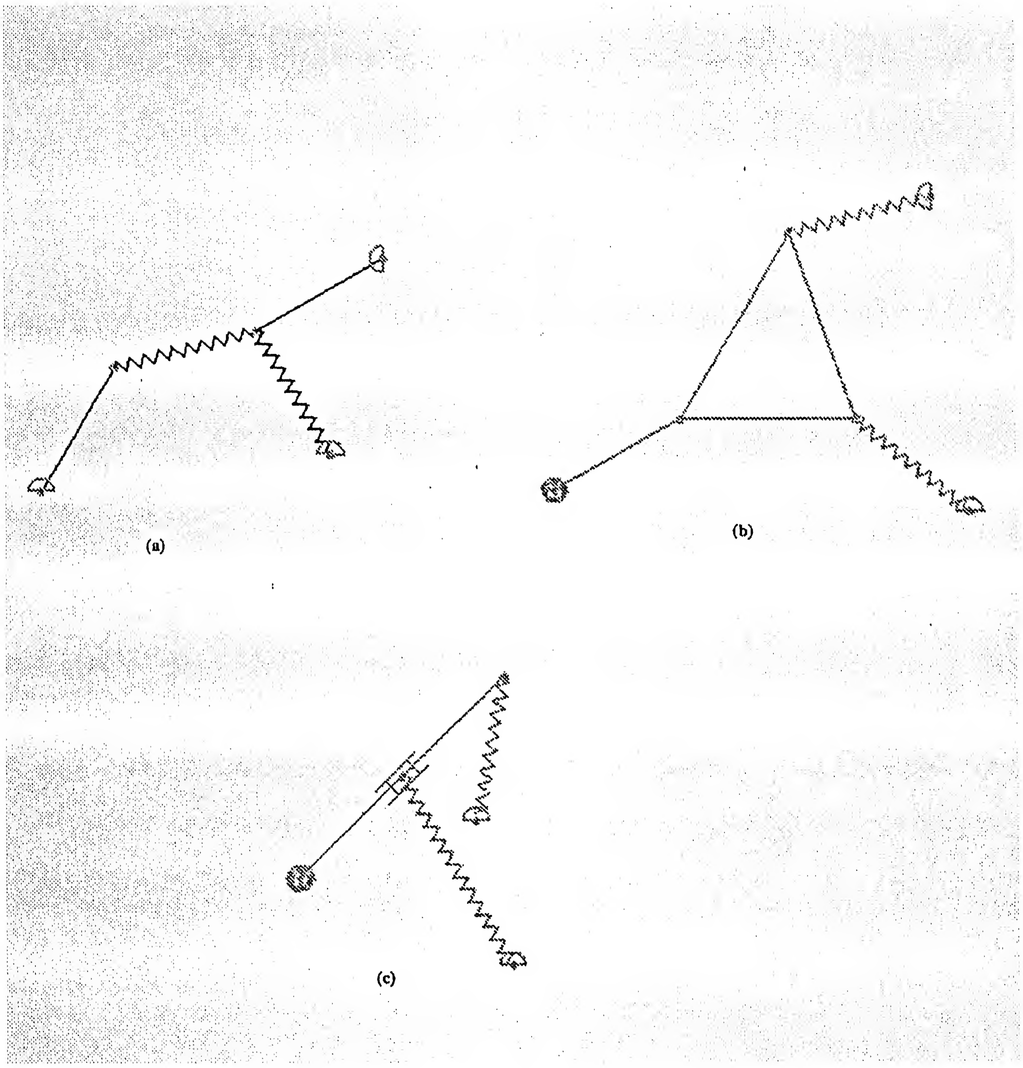


Figure 1.1: Mechanical systems with compliant members

1.1.1 Advantages of computer simulation

Computer simulates the behaviour of various types of multibody systems in great detail during design stages, from first design concepts to the final prototypes. At any stage of design process, **CAA** provides an important amount of data to study the influence of the different design parameters, even before the construction of any physical device or prototype. These tools allow the designer to carry out a large number of simulations quickly and economically. Thus mechanical system analysis can be made more useful to the designer by having the computer carry out repetitive calculations. The engineer can then concentrate on more creative concepts/aspects of the design process.

1.1.2 Requirements

A general purpose computer program for the analysis of mechanical systems must perform following basic functions.

- Accepting data from the user.
- Generating the governing equations of motion.
- Solving the equations automatically.
- Communicating the result to the user.

1.2 Static Equilibrium Analysis Problem

Under the name "*position problem*" fall several different, although closely related, problems.

1.2.1 Initial position problem

The initial position problem is concerned with the determination of the positions of all links of the mechanism, knowing the position of the input links (as many as there are degrees of freedom in the mechanism). Solution of the position problem is important because often it is a preliminary condition

for kinematic and dynamic analysis. In this thesis, this problem is referred to as *assembly* phase, where from approximate input given by user, correct configuration of the system is calculated by using optimisation technique.

1.2.2 The problem of finite displacement

This problem is closely related to the former problem and it lies in finding the displacement of the different links for certain displacements of the input links. The problem can be approached by the same nonlinear techniques as the initial position problem.

1.2.3 Static equilibrium position problem

Finally, this problem determines, as its name indicates, the equilibrium position of the mechanical system subjected to known external forces, including the ability to store potential elastic energy in its compliant members. Fig. 1.2 shows initial configuration of spring restrained slider crank mechanism in thin lines. After applying force F, equilibrium configuration of system is shown in thick lines.

Unlike velocity and acceleration analysis, above mentioned position problem is strongly nonlinear and for this reason, has always presented much greater difficulties for its solution. Kinematic and dynamic analysis of complex mechanical systems is often initiated from a position of static equilibrium. Assigning correct values to the coordinates that describe a state of the static equilibrium, can be a complicated task for a large-scale inter-connected systems of bodies.

1.3 Literature Review

Some methodologies suggested to solve static equilibrium problems are discussed in this section.

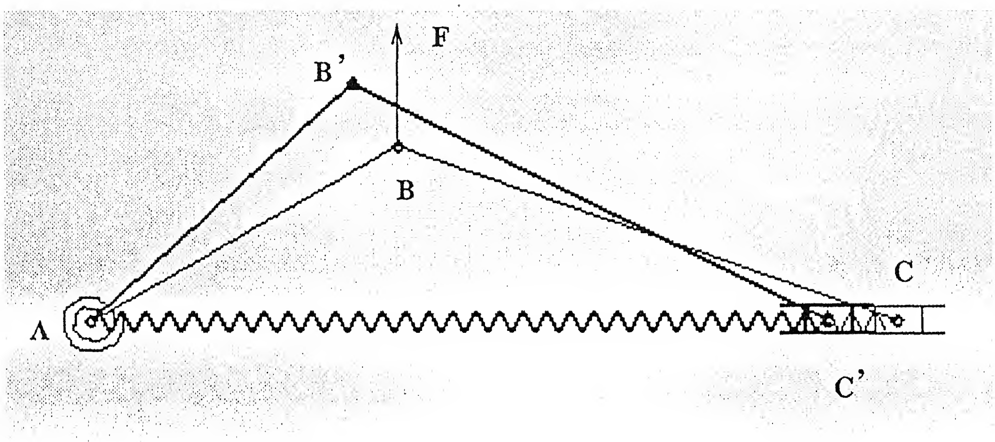


Figure 1.2: Static equilibrium for simple mechanical system

1.3.1 Minimum P.E. and structural strain method

The work of Livermore [1] is considered classic in this field. He describes two methods developed by himself and by Gupta [2], called, respectively, "*Minimum Potential Energy Configuration Method*" and "*Structural Strain Method*". The first process is slow and laborious, especially when the system has several degrees of freedom. The second method begins with a known equilibrium position at which the mechanism has no external forces applied. Then, an incremental process begins, in which external forces are applied little by little and the geometry of the mechanism changes as the forces are applied. This method is based on the analysis of displacement and velocities. Livermore points out that the calculation time according to the second method is usually between 1/2 and 3/4 the time required by the first. But it requires more complex code and considerably greater storage capacity. Also the method may converge to unstable equilibrium, so the stability of solutions must be checked.

1.3.2 Gas method

Most attractive features of this method [3, 4] are conceptual simplicity and the ease with which it can be programmed. It offers a coherent unified way to approach kinematic, static equilibrium, kinetostatic and dynamic problems. It uses basic coordinates. These are the coordinates of some previously chosen points of the mechanical system. The problem is solved like a problem of mathematical programming, and a minimisation method is proposed.

1.3.3 Equilibrium position from equations of motion

Alternative method where static equilibrium position is achieved by setting velocity and acceleration vectors to zero in equations of motion is suggested in [5, 6]. But resultant algebraic equations are highly nonlinear and initial estimates of some of the unknowns, such as Lagrangian Multipliers are not readily available. Also poor estimate of Lagrangian Multipliers may lead to badly conditioned matrix and divergence of the algorithm.

1.3.4 Force equilibrium approach

The basic methodology of this approach is described in [7]. In it's detailed implementation [8], the work is divided into two parts. In the first part, on the basis of approximate information about the system, provided by the user, the initial correct configuration of the system is obtained. In the second part, the equilibrium analysis of the system under the given loading is carried out to get the final equilibrium configuration.

The basic methodology used is to develop the governing constraint equations for rigid and deformable members. These equations are then used in the force equilibrium equation to get the deformed configuration of the system at equilibrium. In this approach, generalised coordinates used are the relative coordinates of every link with respect to the previous link in the loop. Deformable members are modeled as linear and torsion springs.

Although above suggested method is computationally fast, accurate and efficient, it has serious limitations, since it only deals with statically determinate systems, which severely restricts its application area. In this thesis,

the above approach of modeling the system using relative coordinates is used to develop a general formulation which can analyse statically indeterminate systems also.

1.3.5 Potential function approach

In this approach [9], modeling of mechanical system is based on construction of a simple F.E.M. with truss elements. For solution of the problem, error function (Φ) is the potential function of the mechanical system. This function is the sum of two terms, the elastic potential of the deformable elements and the potential of the set of conservative external forces transformed to the nodes of the model. The error function is subjected to constraints that lengths of the rigid elements remain constant. Solution is sought through various optimisation schemes and the methodology covers all position problems.

1.4 Scope of the Present Work

The thesis deals with the static equilibrium of user defined planar and spatial mechanical systems with compliant members. The mechanical system can be statically determinate or statically indeterminate.

After the user provides approximate input of the system, correct initial configuration is obtained by imposing constraint equations. Then equilibrium analysis of the system for given loading is carried out using minimisation of potential energy.

1.5 Assumptions

- Compliant members are considered as linear elastic elements, i.e. there is a linear relationship between the forces applied and the deformation of these members, irrespective of magnitude of forces and deformation.
- Only binary flexible links are considered. Rigid links can have any number of connections.

- Friction due to relative motion of links at joints is neglected. Also the damping present at joints and in the links is neglected.

- Joint clearances are neglected.

- The loading applied on the system is time independent and stationary, i.e. the absolute position of the forces remain the same during the deformation.

1.6 Organization of the Thesis

- Chapter 2 : It introduces general formulation of the problem.

- Chapter 3 : Modeling of systems with various joints is described in detail in this chapter.

- Chapter 4 : It describes *assembly* and *static equilibrium analysis* procedure.

- Chapter 5 : In this chapter, some representative results of assembly and static equilibrium analysis of planar and spatial mechanical systems are reported to establish the capability of the method.

- Chapter 6 : Conclusions and future scope is presented in this chapter.

Chapter 2

Formulation

This chapter introduces various types of generalised coordinates with their relative comparison. In later part of the chapter, general formulation of static equilibrium problem is explained. The methodology is based on [7, 8, 9].

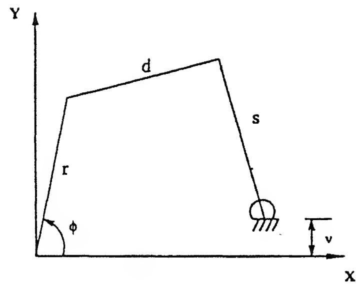
2.1 Generalised Coordinates

Any set of variables that uniquely specifies the position and orientation of all bodies in the system, that is, the configuration of the system, can be taken as the set of *generalised coordinates.* The computational efficiency of a general-purpose program depends upon several factors, two of which are the choice of coordinates and the method of numerical solution. The choice of coordinates directly influences both the number of equations of motion and their order of nonlinearity.

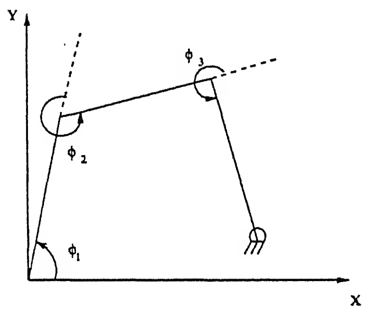
2.2 Selection of Generalised Coordinates

In order to show how different sets of coordinates can lead to different formulations describing the same system, a simple example is given here [5]. In this example, the four-bar linkage is considered for kinematic analysis.

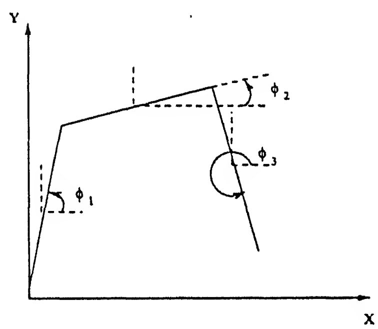
The first formulation shown here considers only one coordinate to describe the configuration of the system. This is referred to as the *minimal coordinate* of the system. As shown in the Fig. 2.1(a), the angle ϕ, describing the orientation of the crank with respect to the X axis, can be selected as the



(a) Minimal coordinate



(b) Relative coordinate



(c) Cartesian coordinate

Figure 2.1: Selection of generalised coordinates

generalised coordinate. For any given configuration, i.e. known ϕ, any other information on the position of any point in the system can be calculated [1].

The second method considers three coordinates ϕ_1, ϕ_2, ϕ_3. Fig. 2.1(b) shows the arrangement. For this mechanism, the selected coordinates define the orientation of each moving body with respect to previous body. Therefore, these coordinates are referred to as *relative coordinates.* Since the four-bar linkage has only one degree of freedom, the three coordinates are not independent. Two loop equations relating these coordinates can be written as follows.

$$r\cos(\phi_1) + d\cos(\phi_1+\phi_2) + s\cos(\phi_1+\phi_2+\phi_3) - l = 0$$
$$r\sin(\phi_1) + d\sin(\phi_1+\phi_2) + s\sin(\phi_1+\phi_2+\phi_3) - v = 0$$

The third formulation uses three *Cartesian coordinates* per link, the x and y coordinates of the center point of each link and the angle of the link which is measured with respect to the X axis. The arrangement is as shown in the Fig. 2.1(c). So there are total nine coordinates dependent upon each other through eight equations.

The crude but general comparison between these three sets of coordinates, with regard to several crucial and important aspects, is summarized in table 2.1 [2]. A general conclusion that can be made from this table is that the smaller the number of coordinates and equations, the higher the order of nonlinearity and complexity of the governing equations of motion, and vice versa. From above comparison it is clear that a set of relative coordinates falls in the middle of the "comparison scale". Therefore the selection of a set of relative coordinates is a good compromise between other two extremes, and the same is used in our formulation. It is also useful in getting derivatives of the governing equations in a simplified way. These derivatives are required in analysis.

[1] upto the branch

[2] Included from reference [5]

	Minimal coordinates	Relative coordinates	Cartesian coordinates
Number of coordinates	Minimum	Moderate	Large
Number of second-order differential equations	Minimum	Moderate	Large
Number of algebraic constraint equations	None	Moderate	Large
Order of nonlinearity	High	Moderate	Low
Derivation of the equations of motion	Hard	Moderately hard	Simple
Computational Efficiency	Efficient	Efficient	Not as efficient
Development of a general-purpose computer program	Difficult	Relatively difficult	Easy

Table 2.1: **Comparison of generalised coordinates**

2.3 Constraint Equations

In mechanical system, a kinematic pair imposes certain conditions on the relative motion between the two bodies it comprises of. When these conditions are expressed in analytical form, they are called *equations of constraint.* A constraint can be seen as any condition that reduces the number of degrees of freedom in a system. The algebraic equations of constraint must be derived such that they are equivalent to physical joints. This equivalence is important and constraint equations must imply the geometry of the joint. In this step, the constraints (if any) among the generalised coordinates are developed in the form as shown below.

$$f(X) = 0 \qquad f \epsilon R^m \tag{2.1}$$

Where, m = number of rigid constraints

i.e.

$$f_1(\phi_1, \phi_2, \phi_3, ...\phi_n) = 0$$

$$f_2(\phi_1, \phi_2, \phi_3, ...\phi_n) = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$f_m(\phi_1, \phi_2, \phi_3, ...\phi_n) = 0$$

Where, $X = [\phi_1, \phi_2, \phi_3, ...\phi_n]^T$ is the set of generalised coordinates.

2.4 Static Determinacy

As the mechanical system under consideration is constrained and its mobility is owing to the deformable members only, we have

$$m + p \geq n$$

Here p is number of deformable coordinates and m is number of rigid constraints.. The cases for which the equality holds are statically determinate cases and those for which the strict inequality holds are the statically indeterminate cases. Fig. 2.2(a) shows a statically determinate system comprising of three rigid and two deformable members. In Fig. 2.2(b), one additional torsional spring at base A, make the configuration statically indeterminate. In a planar system, every link has three degrees of freedom. In Fig. 2.2(a), there are three rigid links, of which one is ground link. Hence, for remaining two links we have $n = 6$. Also $p = 2$, due to two deformable members (links 2 and 3). For constraints, we have to consider various joints in the system. Every revolute joint connecting two rigid links gives two constraints. Due to two revolute joints (R_1 and R_2), m is 4. Since $m + p = n$, this is statically determinate system.

In Fig. 2.2(b), due to additional torsion spring, p becomes three. In this case ($m + p > n$), and system is statically indeterminate. Similarly, the mechanical system shown in Fig. 2.2(c) is also statically indeterminate.

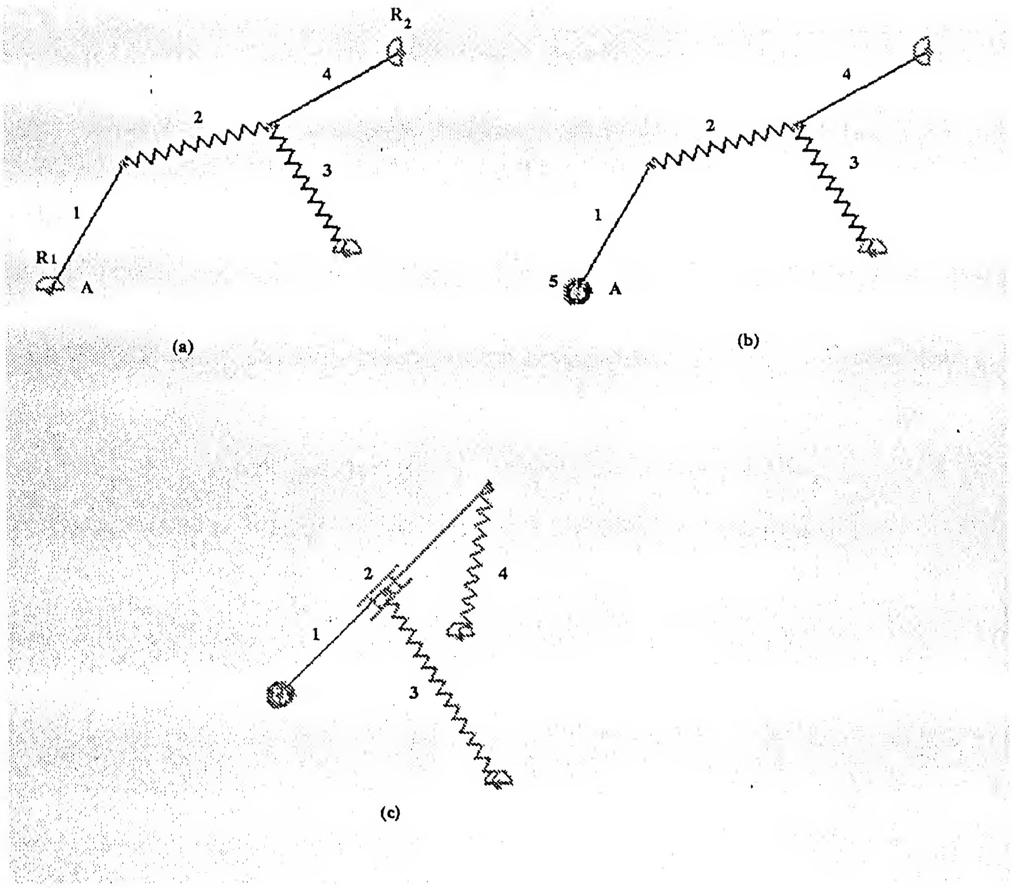


Figure 2.2: Statically determinate and indeterminate cases

2.5 Generalised Force

For the static equilibrium, forces are considered to be acting at the fixed locations and along the fixed orientations, with respect to the global reference frame, throughout the deformation of the system. The loading constitutes of the external stationary forces and moments acting on the system. Weights of various links can also be considered as a part of loading. Input force is transformed to generalised force by calculating Jacobian.

2.6 Static Equilibrium Analysis

Here static equilibrium configuration of the mechanical system is determined by evaluating the position for which the potential energy (P.E.) function is at its minimum. The factors constituting the system P.E. are, gravitational field, externally applied forces and compliant members.

We have used Φ as the P.E. function of the mechanical system. This function is the sum of two terms,

- the elastic potential of the deformable elements and
- the potential of the set of conservative external forces transformed to the nodes of the system.

$$\Phi(\{X\}) = \frac{1}{2}\sum_{i=1}^{a} k_i(d_i(\{X\}) - D_i)^2 + \frac{1}{2}\sum_{j=1}^{b} k_{tj}(\Delta\theta)^2 - \{F\}^T\{\Delta X\} \qquad (2.2)$$

Here,
* a is the number of linear springs
* d_i is the length of the elastic element i in a given iteration
* D_i is the length without deformation
* k_i and k_{tj} are stiffnesses of linear and torsional springs respectively
* $\{F\}$: generalised force
* b is the number of torsion springs
* $\Delta\theta$ is the angular deformation of torsion spring
* $\{X\}$ is the generalised coordinate vector

However, since the mechanical systems can include both rigid and elastic elements, it is necessary that the lengths of the rigid elements remain constant. In other words certain constraints must be satisfied, as discussed in section 2.3. The number of constraints is decided by factors such as whether the system is planar or spatial, number of loops, and type of joint at the end of the loop. Minimisation of potential function in Eq. 2.2, subject to constraints in Eq. 2.1, gives the final equilibrium configuration.

Chapter 3

System Modeling

A closed loop mechanical system may contain one or more loops. Present work can analyse single as well as multiloop systems. Every loop starts from the ground connection of the link at which global reference frame is fixed and ends at the another ground connection. Position of other ground connections in global reference frame should be fixed.

3.1 System Input

The set of input required for describing the mechanical system is :

- Nature of the system (*planar* or *spatial*)
- Total number of links in the system
- Number and type (*linear* or *torsion*) of deformable members
- Connectivity of links in a loop
- Link dimensions
- Ground connection coordinates with respect to global reference frame
- External loading (*forces, moments*) with their location and orientation
- Type of joint at ground connection

- Approximate values of the relative coordinates of the links with respect to previous link along the loop .

The order in which this input is to be provided by the user, is given in the *appendix* (A), for a sample problem. It is important here, to note that the user has the freedom to enter approximate values of generalised coordinates (g.c.) only. The other input parameters such as link lengths and ground connection coordinates must be provided as accurately as possible.

3.2 Link and Joint Modeling

The mechanical system is modeled using concatenated homogeneous transformation matrices as discussed in Craig [11]. These matrices represent geometric relation of every joint in the system.

For every link, local coordinate system is attached. Fig. 3.1 shows two links with their coordinate systems. They are connected by revolute joint. The x-axis of every local coordinate system is along the direction of connection between the two links. The links revolve about the z-axis.

3.2.1 Revolute joint

It offers one degree of freedom. For a planar case shown in Fig. 3.1, link i and revolute joint ij are described by using translation of $L = [l_x, l_y, l_z, 1]$, from local reference frame of link i to that of link j. Subsequently, there is rotation about z axis through angle ϕ. Therefore, the relation of the j-th link with respect i-th link is

$$T_{rev} = T(L) \times R(\phi)$$

$$= \begin{bmatrix} 1 & 0 & 0 & l_x \\ 0 & 1 & 0 & l_y \\ 0 & 0 & 1 & l_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

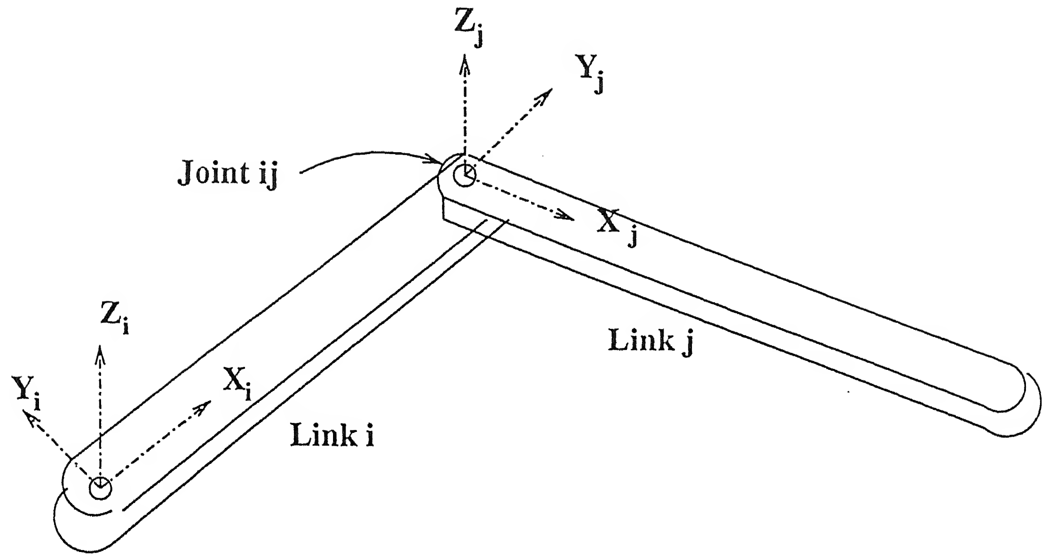


Figure 3.1: **Revolute joint** ij

$$T_{rev} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & l_x \\ \sin\phi & \cos\phi & 0 & l_y \\ 0 & 0 & 1 & l_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this case $l_x = l_y = 0$. Thus, $T_{rev}(L, \phi)$ is the homogeneous transformation, describing the revolute joint. L is the translational distance and ϕ is the amount of rotation with respect to previous link. With this transformation matrix we can express the coordinates on the j-th link with respect to the i-th link.

In case of spatial mechanical systems with revolute joint, the procedure is as follows [10]. Suppose (a, b, c) are direction cosines of axis and (x_1, y_1, z_1) is one point on it. First we reposition the axis so that it passes through the origin. Transformation matrix for it is given by,

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, we make rotation axis parallel to Z axis by first rotating it about X

axis and then about Y axis.

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here,

$$d = \sqrt{b^2 + c^2}$$

and

$$\sin(\alpha) = b/d \qquad \cos(\alpha) = c/d$$

Now, we have aligned the rotation axis with the positive Z axis. The specific rotation angle θ can now be applied as a rotation about the Z axis. To complete the required rotation about the given axis, we need to transform the rotation axis back to its original position. This is done by applying inverse of above transformations. The transformation matrix of rotation for revolute joint then can be expressed as the composition of the several individual transformations :

$$T_{rev} = T^{-1} * R_x^{-1}(\alpha) * R_y^{-1}(\beta) * R_z(\theta) * R_y(\beta) * R_x(\alpha) * T \tag{3.1}$$

3.2.2 Prismatic joint

Here relative displacement between the two links connected is ΔL along z axis. The procedure is same as revolute joint, except that $R_z(\theta)$ in Eq. 3.1 is replaced by $T_{pr}(\Delta L)$ as shown below.

$$T_{pr}(\Delta L) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta L \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2}$$

3.2.3 Cylindrical joint

It offers two degrees of freedom, translation as well as rotation along specified axis. Let ΔL be relative displacement between links and ϕ is relative angle between them. Then

$$T_{cyl}(\Delta L, \phi) = T(\Delta L, 0) \times R(0, \phi)$$

$$T_{cyl}(\Delta L, \phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta L \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

In case of spatial mechanical systems, matrix $R(0, \phi)$ is calculated similar to the procedure explained in revolute joint.

3.2.4 Spherical joint

It allows three degrees of freedom, i.e. rotation around all the three axis of one link relative to another. Here the Z-Y-X *Euler angles* are used to describe rotations. So,

$$T_{sph}(L, \phi_z, \phi_y, \phi_x) = T(L, \phi_z) \times R(\phi_y) \times R(\phi_x) \tag{3.4}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi_z & 0 & L \\ \sin\phi_z & \cos\phi_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi_y & 0 & \sin\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi_y & 0 & \cos\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_x & -\sin\phi_x & 0 \\ 0 & \sin\phi_x & \cos\phi_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3.2.5 Screw joint

It is similar to the cylindrical joint, except translation and rotation are related through pitch of screw., The screw joint is represented by,

$$T_{sc}(k, \phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & L + \frac{k\phi}{2\pi} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

Where, ϕ is the relative angle between the links, and k is lead of the screw.

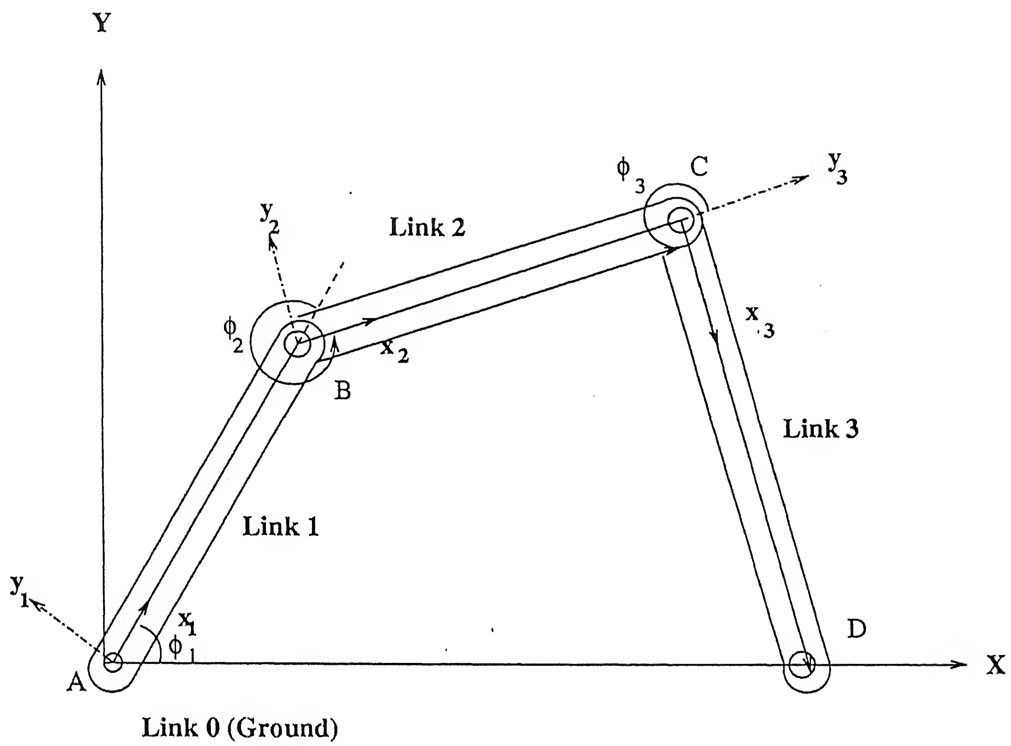


Figure 3.2: **Four bar mechanism**

3.3 Loop Closure Equations

The governing equations are developed using loop closure equations. It consists of concatenation of homogeneous transformation matrices, along each loop in the system. A four-bar mechanism is used as an example to explain the loop closure equation. It is shown in Fig. 3.2. Here, (X,Y,Z) is the global reference system. Links 1, 2 and 3 have local coordinate systems $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) respectively, fixed to them. Consequently, the (x_1, y_1, z_1) coordinate system is defined relative to the global coordinate system by

$$T_1 = T(0, \phi_1)$$

where ϕ_1 is defined in the system 0. Furthermore, the (x_2, y_2, z_2) coordinate system is defined relative to the (x_1, y_1, z_1) system by

$$T_2 = T(L_1, \phi_2)$$

where L_1 and ϕ_2 are defined in system 1. In a similar fashion,

$$T_3 = T(L_2, \phi_3)$$

where L_2 and ϕ_3 are defined in system 2. However, this does not close the loop for the mechanism. But, this matrix multiplication is sufficient for getting constraints, since ground connection is reached. The accurate location of end ground connection in this loop is already known. Let it be $D = [x, y, z, 1]^T$. Then, constraint equations can be written as

$$f = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\} - \{D\} = \{0\} \qquad (3.6)$$

This gives two constraint equations (for x and y), in case of planar mechanical systems. Also ϕ_1, ϕ_2, ϕ_3 being the generalised coordinates, these constraints are expressed as functions of them.

Jacobian calculation, which needs differentiation of concatenated transformation matrices, is made easier by use of relative coordinates. It can be observed that, any one g.c. of the system described by loop closure equation, is wholly contained in only one of the matrices making up the product. Thus, while taking the derivatives of loop closure equations with respect to a g.c., say ϕ_1, only the components in the matrix having ϕ_1 are changed and the rest of the matrix product remains unchanged.
Thus, we have,

$$\begin{aligned}\frac{\partial f}{\partial \phi_1} &= \frac{\partial}{\partial \phi_1}[T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\}] \\ &= T'(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\} \qquad (3.7)\end{aligned}$$

Similarly, the derivatives with respect to all the g.c. can be calculated.

Chapter 4

Static Equilibrium Analysis

Analysis is done in two stages. In the first part, from the approximate input provided by the user, correct initial configuration is obtained. It satisfies all the constraints. This provision is useful in case of complicated planar and spatial systems, where giving accurate input system data is very difficult. In the next phase, static equilibrium analysis of the assembled configuration is performed and the final configuration is displayed in graphical format.

The entire procedure is explained using six link planar system as shown in Fig. 4.1. It has rigid as well as deformable links. Whenever the system differs from this example (as in spatial cases), the difference and the relevant analysis is explained at suitable places.

4.1 System Generalised Coordinates

There are two loops in the system, with connectivity 1-2-3 and 1-2-4. Torsion spring is not considered as part of loop. Along the loop 1, link 1 is connected to the ground (base). Only one orientation is sufficient to define link 1 with respect to the base. Hence, ϕ_1, i.e. the relative angle between the link 1 and base, in the reference system of link zero (ground) is g.c. associated with it. Since all joints are revolute joints, g.c. associated with other links are as follows :

* link 2 - ϕ_2, relative angle between the links 1 and 2
* link 3 - ϕ_3, relative angle between the links 2 and 3

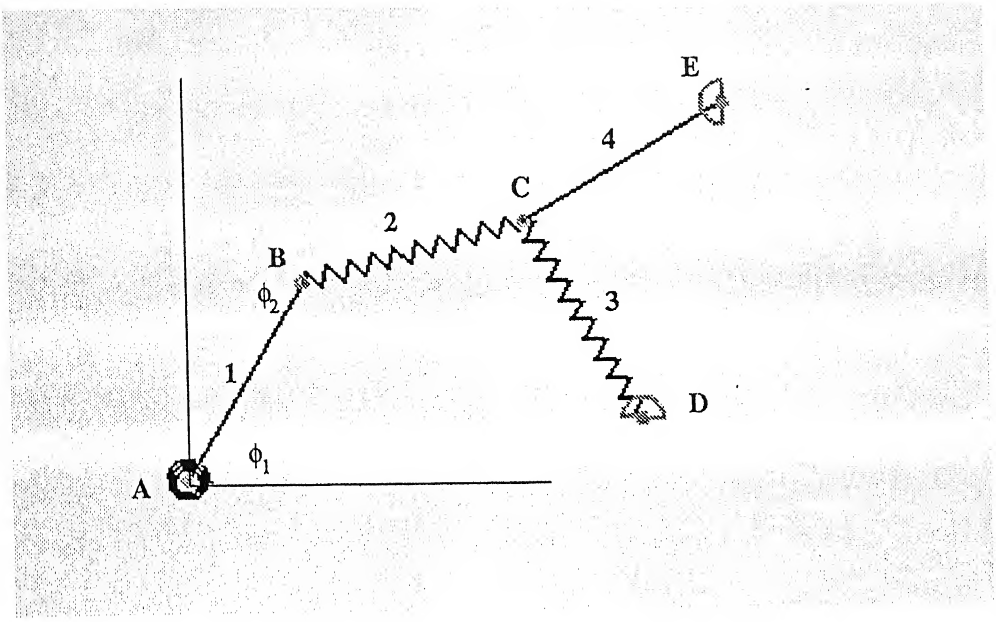


Figure 4.1: **Mechanical system with three deformable links**

* link 4 - ϕ_4, relative angle between the links 2 and 4
For this system, g.c. vector is

$$\{X\} = [\phi_1, \phi_2, \phi_3, \phi_4]^T$$

The number of g.c. for a link is decided by the type of joint with which the link is connected with the previous link. The g.c. for the links connected by various joints (with their previous links), are as given below :

- Revolute joint - 1 (orientation with respect to previous link),
- Prismatic joint - 1 (displacement with respect to previous link),
- Spherical joint - 3 (3 Euler angles with respect to previous link),
- Cylindrical joint - 2 (orientation and displacement with respect to previous link),
- Screw joint - 1 (orientation with respect to previous link)

4.2 Assembly Procedure

As discussed earlier, providing the correct initial relative coordinates, for the complex planar and spatial mechanical systems is a tough task for the user. To help him through this, the assembly phase is used. Optimisation technique is used to correct the configuration. Fig. 4.2 shows initial and corrected configuration for a simple system. All the links are treated as rigid links, since there is no deformation at this stage of analysis. The entire procedure is explained in this section.

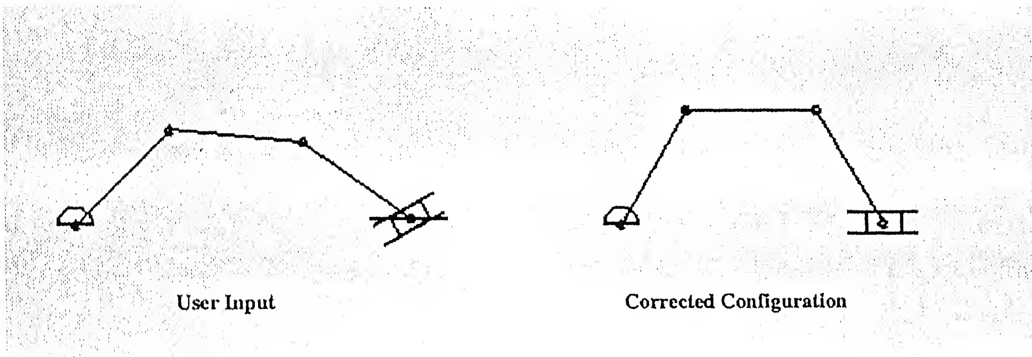


Figure 4.2: **Correction of user input using assembly**

4.2.1 Joint modeling

First, from the data provided by the user, various joints in the loop are modeled using transformation matrices (refer section 3.2). These individual matrices are then concatenated for each loop, so that global coordinates of ground connections can be calculated. Using the loop closure equations, the coordinates of joints D and E (Fig. 4.1), in the global reference frame are calculated as,

$$\{D\} = \left\{ \begin{array}{c} D_x \\ D_y \\ D_z \\ 1 \end{array} \right\} = T(0,\phi_1) * T(L_1,\phi_2) * T(L_2,\phi_3) * \{L_3\} \tag{4.1}$$

and

$$\{E\} = \begin{Bmatrix} E_x \\ E_y \\ E_z \\ 1 \end{Bmatrix} = T(0,\phi_1) * T(L_1,\phi_2) * T(L_2,\phi_4) * \{L_4\} \tag{4.2}$$

These coordinates are needed in formulating constraint equations.

4.2.2 Constraint equations

Number of constraint equations depend on the type of joint at the end of the loop. Suppose in the planar system revolute joint is at the end. It creates two constraints viz. the global x and y coordinates of the end link at that joint, are same as the global x and y coordinates of the ground connection.

A) Constraint equations for planar systems

Fig. 4.3 shows the system assembled from approximate data provided by the user. Joints D and E are not coincident with the ground connections D' and E'.

- **Revolute joint at the end of loop**

For both the loops, the joints at the ends are revolute. Hence, the constraints are that the points D and D' in loop 1, and E and E' in loop 2 must coincide. In other words, x and y coordinates of these points must be same. Thus, we have,

$$\{D\} = \{D'\} \quad and \quad \{E\} = \{E'\}$$

D and E are calculated in previous subsection. D' and E' are accurate coordinates of ground connection. Now, we get two constraint equations per loop, which are

$$f_1 = D_x - D'_x = 0,$$

$$f_2 = D_y - D'_y = 0,$$

$$f_3 = E_x - E'_x = 0,$$

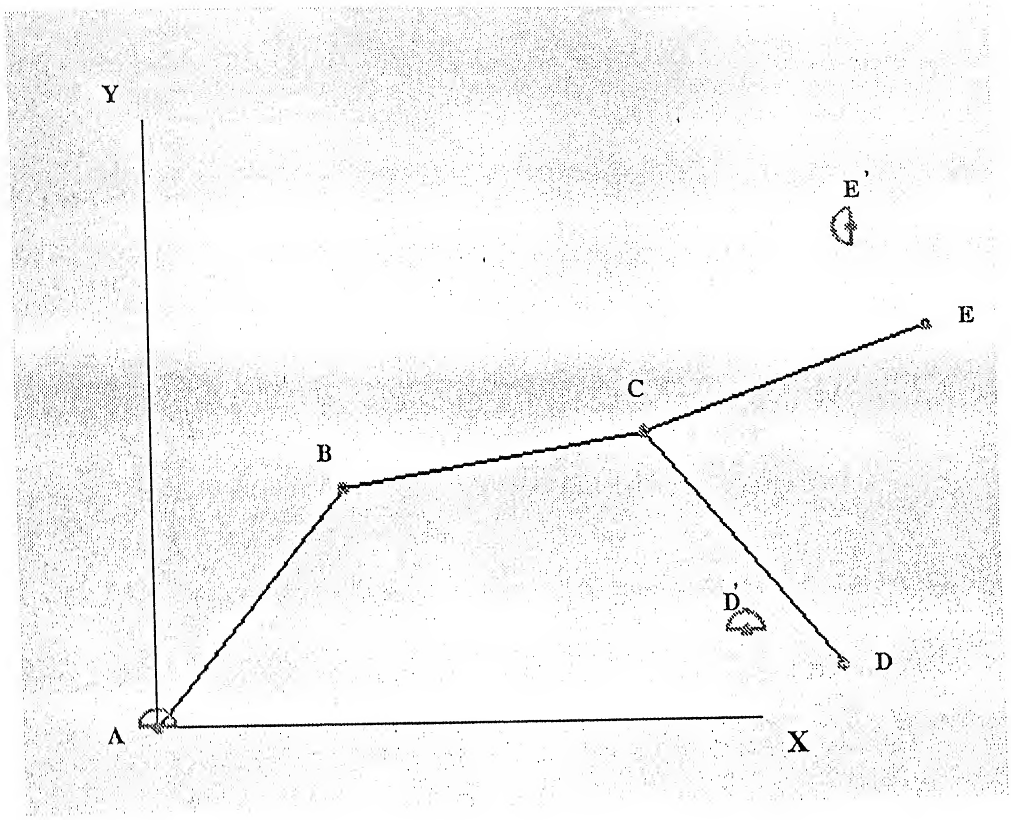


Figure 4.3: **Revolute joint at the end of a loop**

and

$$f_4 = E_y - E_y' = 0;$$

or

$$\left\{ \begin{array}{c} f_1 \\ f_2 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{L_3\} - \left\{ \begin{array}{c} D_x' \\ D_y' \end{array} \right\} \tag{4.3}$$

and

$$\left\{ \begin{array}{c} f_3 \\ f_4 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_4) * \{L_4\} - \left\{ \begin{array}{c} E_x' \\ E_y' \end{array} \right\} \tag{4.4}$$

In this way, four constraint equations are obtained for this planar system (2 equations per loop).

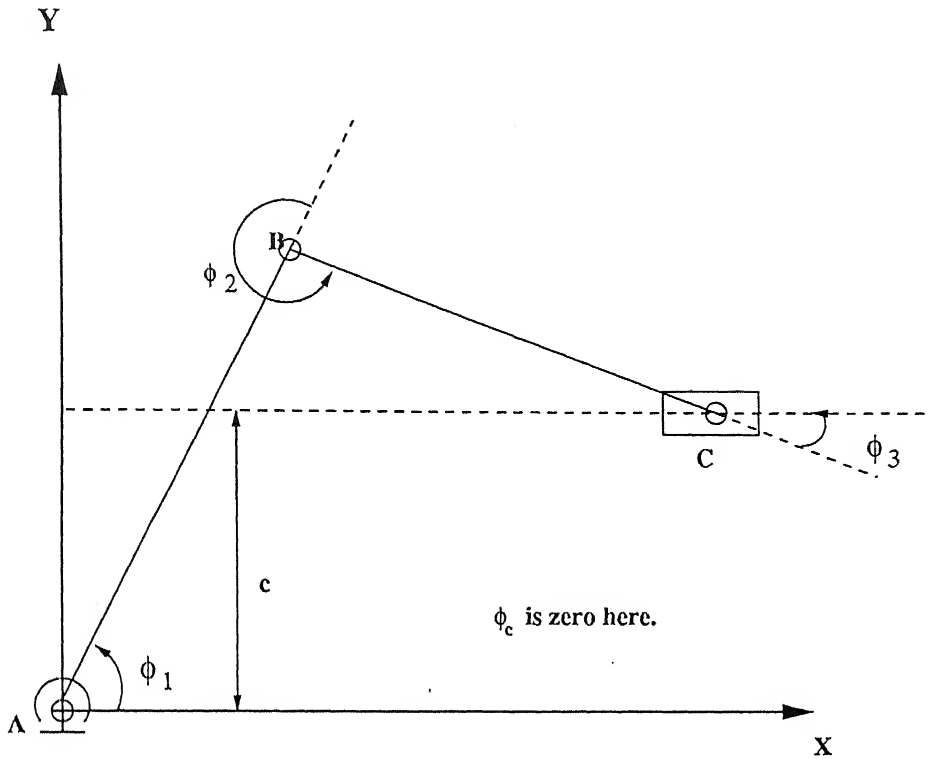


Figure 4.4: **Prismatic joint at the end of a loop**

- **Prismatic joint at the end of loop**

The two constraints due to prismatic joint at the end of loop, are that the slider moves along a fixed straight line and its orientation with respect to Z axis of global reference system is fixed. That is, we have

$$y - mx - c = 0$$

and

$$\phi - \phi_c = 0$$

ϕ_c is orientation of slider axis with respect to X axis. Thus, in the Fig. 4.4 the two constraints obtained for the loop are

$$C_y - mC_x - c = 0 \tag{4.5}$$

and

$$\phi_1 + \phi_2 + \phi_3 - \phi_c = 0 \tag{4.6}$$

B) Constraint equations for spatial systems

In spatial systems, the number of constraints per loop are three or more depending upon the type of joint at the end of loop. Getting the governing constraint equations is a bit complicated than planar systems.

- **Spherical joint at the end of loop**

For a spherical joint at the end, we get three constraints : the global X, Y and Z coordinates of the end link must be same as the global X, Y and Z coordinates of the corresponding ground connection. In Fig. 4.5, position of D is calculated using the loop closure equations. Since the user provides correct coordinates of D, the three constraints are obtained as

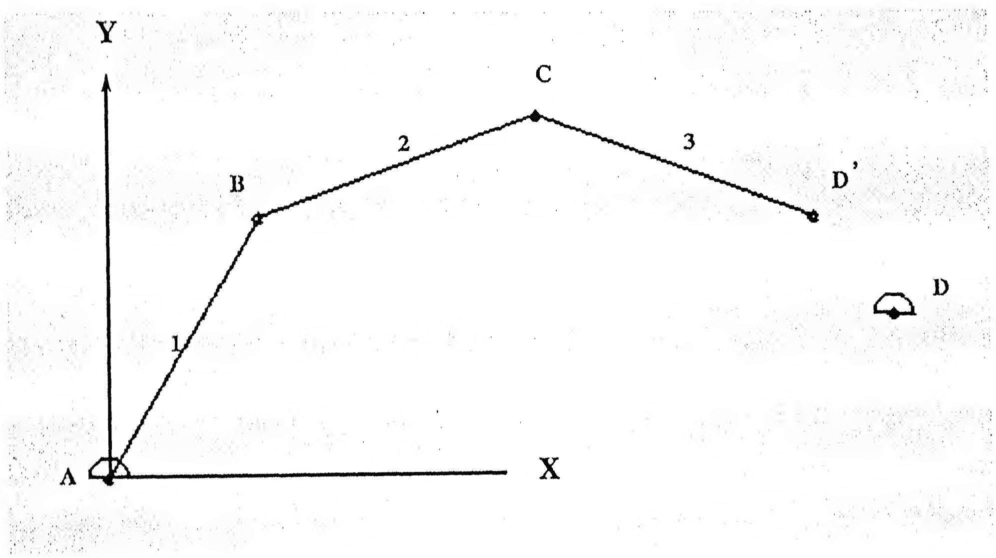


Figure 4.5: Spherical joint at the end of loop

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{Bmatrix} D'_x \\ D'_y \\ D'_z \end{Bmatrix} - \begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} \tag{4.7}$$

- **Revolute joint at the end of loop**

It has one degree of freedom, the orientation about the rotational axis. So, it offers five constraints. Three position constraints are that coordinates of points D and D' should match as shown in Fig. 4.5. They are obtained similar to that of the spherical joint. The remaining two orientation constraints are obtained as follows.

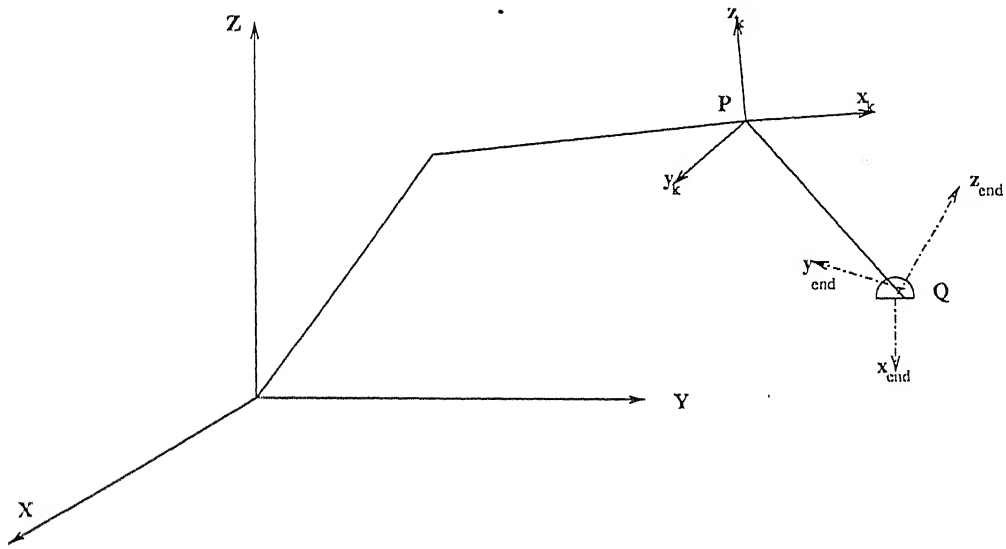


Figure 4.6: **Revolute joint at the end of loop**

R_k is the matrix, we get by traversing the loop from global frame to k-th frame. ${}^kR_{end}$ is the description of end frame with respect to the k-th frame, and R_{end} is the description of the end frame with respect to the global frame. So, we have,

$$R_k \times^k R_{end} = R_{end} \tag{4.8}$$

Initially, R_{end} is fixed with its z axis along revolute axis and x, y axis in a plane perpendicular to it (forming a right-handed triad). During analysis as relative coordinates changes, R_{end} also changes. But, orientation of revolute axis in global frame must be same. So, we get two orientation constraint

equations by comparing third column (for z axis) of initial R_{end} and updated R_{end} during the analysis.

- **Prismatic joint at the end of loop**

A prismatic joint has only one degree of freedom, the translation along the given axis (direction), and it offers five constraints. The position constraints are obtained as follows. Let l, m, n, be the direction cosines of slider axis. Then, its equation is,

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} \tag{4.9}$$

Now, the reference point on slider should satisfy Eq. 4.9, which gives two constraints. Orientation constraints are obtained similar to revolute joint. But, since orientation of slider axis is fixed in global frame, we get three constraints by comparing initial R_{end} and updated orientation matrix R_{end}.

4.2.3 Minimisation of objective function

This is the final step in assembly phase. The constraint equations obtained are a set of nonlinear equations. The solution of these equations will give us the correct initial configuration. Either we can directly solve the set of simultaneous equations,

$$\begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} = \{0\}$$

or, we can pose the problem in such a way that, the sum of squares of all the constraint equations is zero, i.e.

$$f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = 0 \tag{4.10}$$

Eq. 4.10 can be solved as an optimisation problem

$$U = f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 \tag{4.11}$$

with U as objective function to be minimized. Thus, it becomes an unconstrained nonlinear multi-variable optimisation problem. It is solved in MATLAB with function *fmins*. The solution obtained is correct initial configuration. This is analysed for static equilibrium in next section.

4.3 Static Equilibrium Analysis

For static equilibrium analysis, major steps followed are :

- Calculation of lengths of deformable links
- Generalised force calculation
- Objective function formulation
- Minimisation of objective function

Above steps are explained in detail in the following subsections.

4.3.1 Calculation of lengths of deformable links

In static equilibrium analysis, deformation of compliant members must be considered. Here, we need to calculate two link lengths, l_2 in loop 2 and l_3 in loop 1 (Fig. 4.7).

First consider link 2 with length l_2. Suppose, the point B is at (x_1, y_1) and C is at (x_2, y_2), with respect to global reference frame. Fig. 4.7 shows the arrangement. Length l_2 is equal to the distance between the points B and C. Thus,

$$l_2 = \sqrt{x^2 + y^2} \tag{4.12}$$

where $\sqrt{x^2 + y^2}$ is the distance between the joints B and C, approached from both ground connections of the loop in which the flexible member is present. Here,

$$x = x_2 - x_1$$

and

$$y = y_2 - y_1$$

Using the loop closure equation, we get,

$$\left\{ \begin{array}{c} x_1 \\ y_1 \end{array} \right\} = T(0, \phi_1) * \{L_1\} \tag{4.13}$$

and

$$\left\{ \begin{array}{c} x_2 \\ y_2 \end{array} \right\} = \left\{ \begin{array}{c} E_x \\ E_y \end{array} \right\} - T(\phi_1) * T(\phi_2) * T(\phi_4) * \{L_4\} \tag{4.14}$$

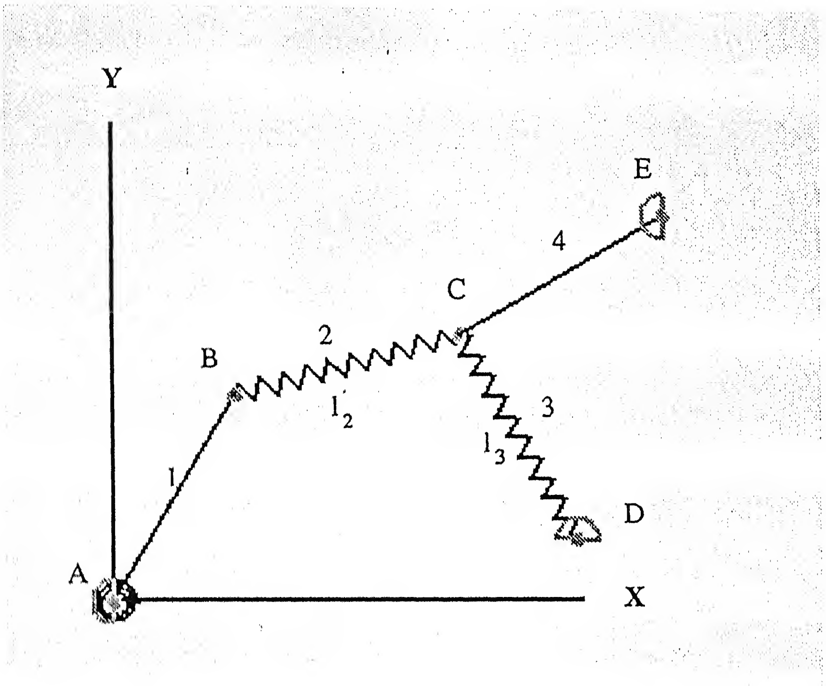


Figure 4.7: **Calculation of length of deformable link**

After calculating (x_1,y_1) and (x_2,y_2), length l_2 can be calculated. In a similar fashion, the link length l_3 in loop 1 will be calculated. The calculation of (x_1,y_1) and (x_2,y_2) is same, irrespective of the type of joint in the loop. The only difference it makes is the concatenation of homogeneous transformation matrices changes depending upon the type of joints along the loop.

In case a deformable member is at the start of the loop, we have, $(x_1, y_1) = (0, 0)$. Similarly, if the deformable member is at the end of loop, the coordinates (x_2, y_2) are the same as the coordinates of the end ground connection of the loop.

4.3.2 Generalised force

The system deforms under the action of external loading. These forces include external forces, torques and body forces i.e. weights of links.

Fig. 4.8(a) shows external loading acting on the system. Force F is acting at a distance L along link 1, from ground connection. It makes angles α_x and α_y with respect to the global X and Y axis respectively. Moment M_1 is acting counterclockwise at the c.g. of link 4. Let W_1, W_2, W_3, W_4 be the weights of links 1, 2, 3 and 4 respectively. First, location of force in a global reference frame is calculated. Let, (x, y) be the location of force with respect to global reference frame and $L = [l_x, l_y, l_z, 1]^T$. Then,

$$\left\{ \begin{array}{c} x \\ y \end{array} \right\} = T(0, \phi_1) * \{L\} \tag{4.15}$$

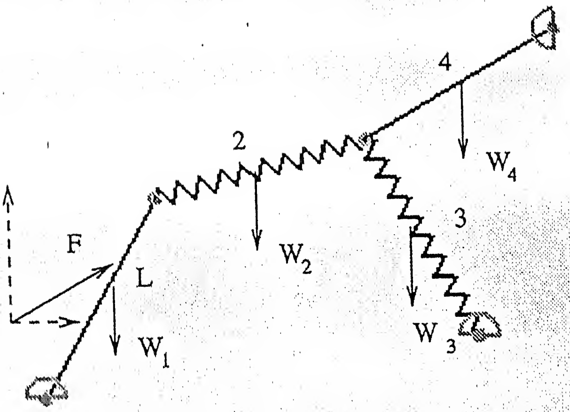
For Jacobian calculation, the above equation has to be differentiated with respect to g.c. The partial derivatives of Eq. 4.15, with respect to ϕ_1, ϕ_2, ϕ_3, ϕ_4 are given by

$$\left\{ \begin{array}{c} \frac{\partial x}{\partial \phi_1} \\ \frac{\partial y}{\partial \phi_1} \end{array} \right\} = T'(0, \phi_1) * \{L\}, \qquad \left\{ \begin{array}{c} \frac{\partial x}{\partial \phi_2} \\ \frac{\partial y}{\partial \phi_2} \end{array} \right\} = \{0\}$$

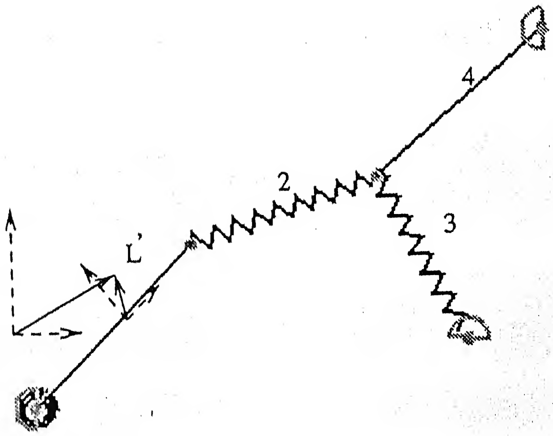
$$\left\{ \begin{array}{c} \frac{\partial x}{\partial \phi_3} \\ \frac{\partial y}{\partial \phi_3} \end{array} \right\} = \{0\}, \qquad \left\{ \begin{array}{c} \frac{\partial x}{\partial \phi_4} \\ \frac{\partial y}{\partial \phi_4} \end{array} \right\} = \{0\}$$

Weights of links are acting at c.g. of respective link. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) be the location of c.g. of every link in a global frame. $\{C_1\}, \{C_2\}, \{C_3\}$ and $\{C_4\}$ are their position vectors, from the previous joints. Then we have,

$$\left\{ \begin{array}{c} x_1 \\ y_1 \end{array} \right\} = T(0, \phi_1) * \{C_1\}$$



(a) Initial Position



(b) Position after iteration 1

Figure 4.8: **Forces acting on system**

$$\left\{ \begin{array}{c} x_2 \\ y_2 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * \{C_2\}$$

$$\left\{ \begin{array}{c} x_3 \\ y_3 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_3) * \{C_3\}$$

$$\left\{ \begin{array}{c} x_4 \\ y_4 \end{array} \right\} = T(0, \phi_1) * T(L_1, \phi_2) * T(L_2, \phi_4) * \{C_4\}$$

The partial derivatives of the above equations with respect to g.c. are calculated similar to the procedure discussed above.

In case of moments, the angle of link on which the moment is acting, is expressed in terms of g.c. Let link 4 make angle ϕ with respect to global X axis. Then

$$\phi = \phi_1 + \phi_2 + \phi_4$$

and

$$\frac{\partial \phi}{\partial x} = [1, 1, 0, 1]$$

The actual force acting on the system, is the set of force components acting along the X, Y and Z directions. Moments are also acting about the c.g. of some of the links. The actual force is denoted by F_{in}.

$$F_{in} = [F_1 \cos \alpha_x, F_1 \sin \alpha_y, 0, -W_1, 0, -W_2, 0, -W_3, 0, -W_4, +M_1]^T$$

Updating the F_{in}

As stated earlier, the positions of the external loads are considered fixed, with respect to the global reference frame, during the whole analysis. Hence, after each iteration, the locations of these forces changes with respect to the links on which they are applied. Due to change in position, Jacobian changes, which is calculated as given below.

Consider the force F whose location in global reference frame is calculated as (x, y). With respect to link 1 it is at $L(p_x,p_y)$. After the first iteration, the system deforms and hence the location of the point with respect to the link 1 changes to (p_x',p_y'). Fig. 4.8(b) shows configuration after iteration 1. Suppose, g.c. for link 1 in this iteration is ϕ_1'. Then

$$T(0,\phi_1') * \{L'\} = \left\{ \begin{array}{c} x \\ y \end{array} \right\}$$

This gives,

$$\{L'\} = [T(0,\phi_1')]^{-1} \left\{ \begin{array}{c} x \\ y \end{array} \right\}$$

Now, the new Jacobian is calculated with position vector L'. Finally generalised force, F_{gen} is given as

$$F_{gen} = J^T * F_{in} \tag{4.16}$$

4.3.3 Objective function

Objective function is the potential function of the mechanical system. The factors contributing to potential function are forces acting on the system and springs. If there are n springs, with stiffness k_i, undeformed length D_i, and length in a given iteration as d_i, then their contribution to potential energy function is given by

$$p_1 = \frac{1}{2}\sum_{i=1}^{n} k_i(d_i(\{x\}) - D_i)^2 \tag{4.17}$$

Torsional springs are assumed to be undeformed at initially correct configuration. Their contribution to potential energy function is given by

$$p_2 = \frac{1}{2}\sum_{j=1}^{n_t} k_{tj}(\theta_j' - \theta_j)^2 \tag{4.18}$$

Here, n_t is total number of torsional springs, k_{tj} is stiffness of j-th spring and $(\theta_j' - \theta_j)$ is angular deformation. Finally, potential energy due to generalised force is

$$p_3 = -F_{gen}^T\{\Delta X\} \tag{4.19}$$

Sum of the three expressions above gives the objective function Φ.

$$\Phi = p_1 + p_2 + p_3 \tag{4.20}$$

4.3.4 Minimisation of objective function

A mechanical system also contains rigid links, whose lengths are constant. Hence, objective function is subjected to constraints as discussed in section 4.2.2. It is a constrained optimisation problem. Minimisation of Φ is carried out by MATLAB function *constr*, which gives the final equilibrium configuration. This equilibrium configuration is then graphically displayed.

Chapter 5

Results and Discussions

The static equilibrium analysis of a few planar and spatial mechanical systems is presented in this chapter.

Presentation scheme is as follows :

- System input data and numerical output is described first.
- Below each system input, mechanical system with external forces is shown.
- In the next figure, result of *assembly* and *static equilibrium* is shown.
- Diagram (a) shows *assembly* output (thick lines) and diagram (b) shows *static equilibrium* configuration in thick lines.
- In dia. (a) incorrect (user-supplied) configuration is superimposed in thin lines, and in dia. (b) initial correct configuration is shown in thin lines for easy comparison.

Example 1 :

* Mechanical system constitutes of 4 rigid and 2 deformable members.
* It is a statically indeterminate system and is shown in Fig. 5.1.
* Link Lengths : $l_1 = 10$, $l_2 = 15$, $l_3 = 22.8024$
* Stiffness : $k_3 = 60$ N/cm $\quad k_5 = 200$ N-cm/rad
* Ground connection coordinates : A(22.8024,0.0)
* Force : 200 N is acting at *revolute joint* J_1, Orientation : 90^0
* Moment : 100 N-cm at link 1
* User input : [40, -60, 30]
* Initial correct configuration : [30.0002, -49.4714, 19.4705]
* Equilibrium Configuration : [42.4762, -68.8051, 26.4288]
* Final spring length $l_3 = 20.8195$ cm
* Torsion spring deformation : 0.2177 rad

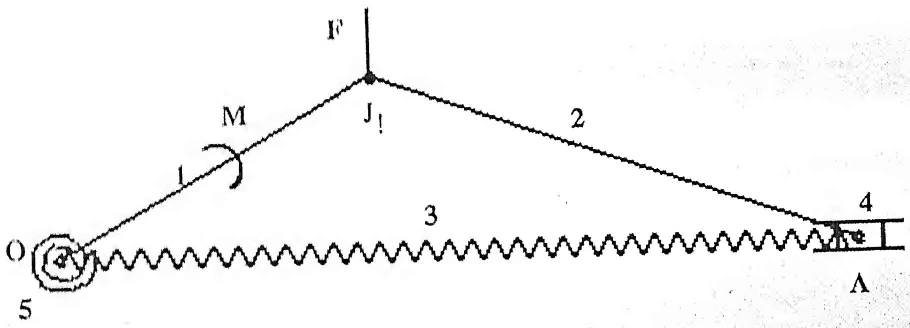
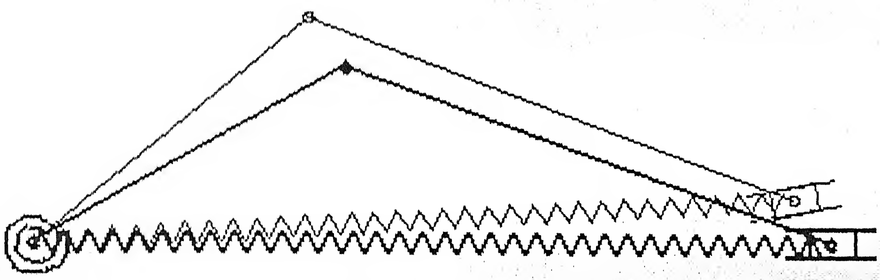
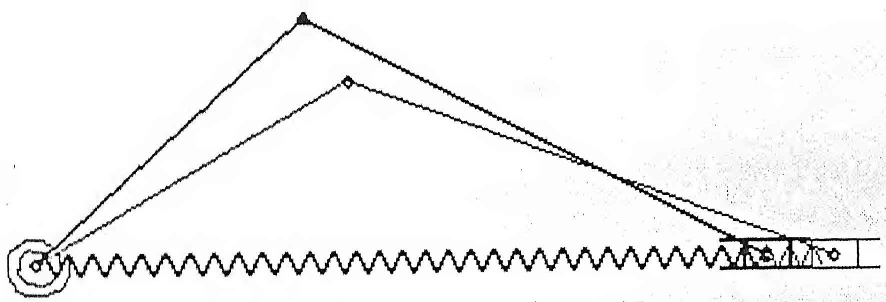


Figure 5.1: Example 1 : Mechanical system



(a) Initial Configuration (Ex. 1)



(b) Equilibrium Configuration (Ex. 1)

Figure 5.2: Example 1 : Results

Example 2 :

* System consists of 6 rigid links and one deformable link.
* It is a statically determinate system and is shown in Fig. 5.3.
* Link Lengths : $l_1 = 10$, $l_2 = 15$, $l_3 = 10$, $l_4 = 15$
* Stiffness of torsion spring : 2000 N-cm/rad
* Ground connection coordinates : A(4.8102, 3.4763), B(23.5364,4.1678)
* Force : 100 N is acting at *slider*, Orientation : 0^0
* User input : [100, -100, -120, -30, 50]
* Initial correct configuration : [114.99, -104.99, -135, -40, 29.99]
* Equilibrium Configuration : [98.44, -103.54, -144.56, -11.79, 15.08]
* Torsion spring deflection = 0.2889 rad

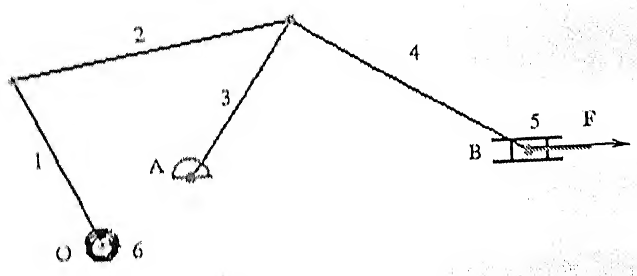
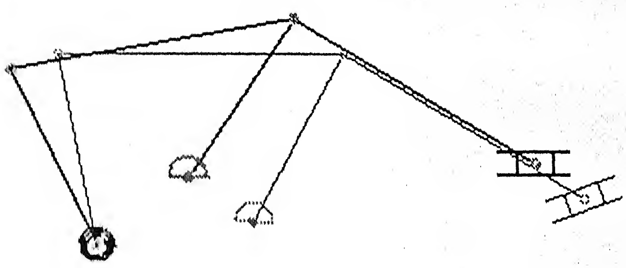
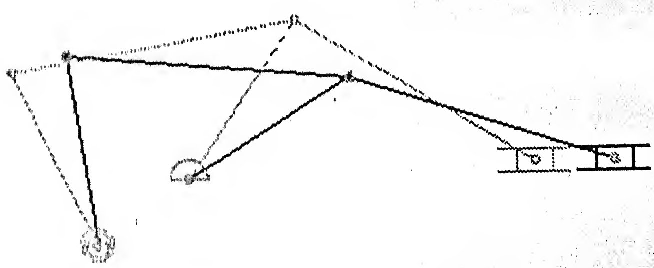


Figure 5.3: Example 2 : Mechanical system



(a) Initial Configuration (Ex. 2)



(b) Equilibrium Configuration (Ex. 2)

Figure 5.4: Example 2 : Result

- Here Example 2 is analysed for three different forces.
- Initial and equilibrium configuration is superimposed in Fig. 5.5

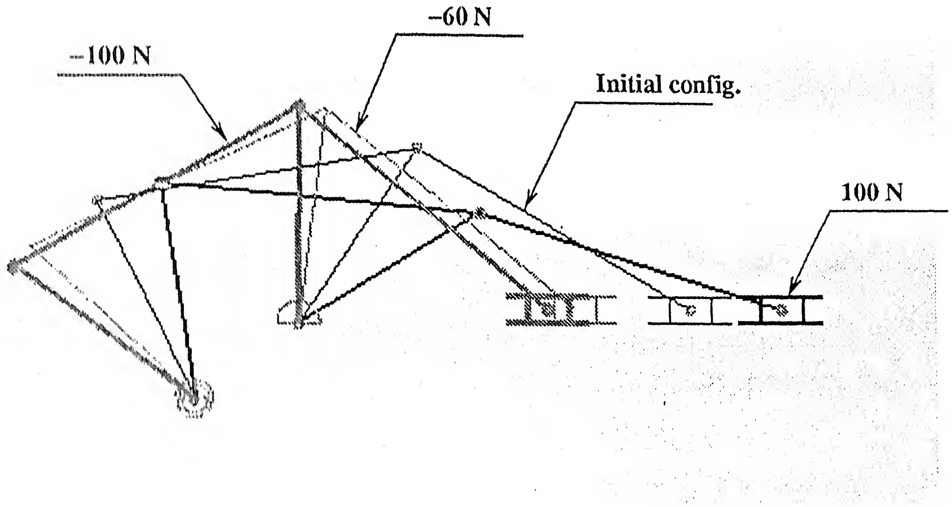


Figure 5.5: Example 2 : Configurations for varying load

- It is observed that increasing force magnitude increases deformation of compliant members.
- Also increasing spring stiffness decreases its deformation (result not shown here).

Example 3 :

* Statically determinate system having 4 rigid and 3 deformable members.
* Link Lengths : $l_1 = 10$, $l_2 = 15$, $l_3 = 9.0$, $l_4 = 12.5$ cm
* Stiffness : $k_3 = k_4 = 60$ N/cm $k_6 = 400$ N-cm/rad
* Ground connection coordinates : A(20.3541,-0.85251), B(33.6717,14.7744)
* Force : 600 N at 4 cm from *revolute joint* J_1, Orientation : 25^0
* Moment : 3000 N-cm at link 1
* Initially slider is at 6 cm along link 2.
* User input : [35, -10, -45, 30]
* Initial correct configuration : [30, -20, -60, 25]
* Equilibrium Configuration : [40.0361, -27.6175, -90, 12.3189, 10.83]
* Final spring lengths : $l_3 = 9.8454$ and $l_4 = 12.4646$ cm
* Torsion spring deformation : 0.17518 rad

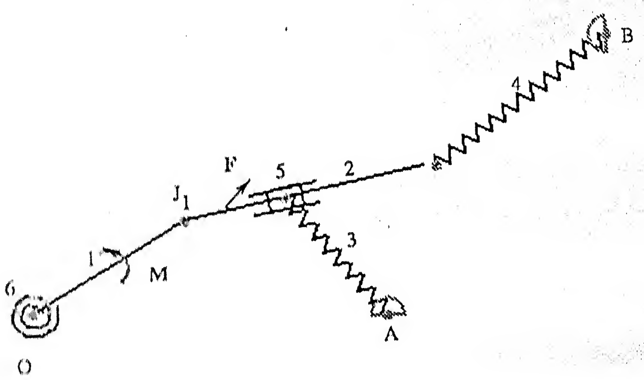


Figure 5.6: Example 3 : Mechanical system

Example 4 :

* Mechanical system having 3 rigid and 3 deformable members.
* It is a statically indeterminate system and is shown in Fig. 5.8.
* Link Lengths : $l_1 = l_2 = l_3 = l_4 = 10$
* Stiffness : $k_3 = k_4 = 70$ N/cm $\quad k_6 = 200$ N-cm/rad
* Ground connection coordinates : A(23.3195,16.2484), B(19.6592,2.5881)
* Force : 150 N at 5 cm from *revolute joint* J_1, Orientation : -60^0
* Moment : 100 N-cm at link 2
* User input : [50, 25, -30, -70]
* Initial correct configuration : [60, 15, -45, -75]
* Equilibrium Configuration : [47.26, 7.35, -22.95, -84.42]
* Final spring lengths : $l_3 = 9.3595$ and $l_4 = 9.7091$ cm
* Torsion spring deformation : 0.22238

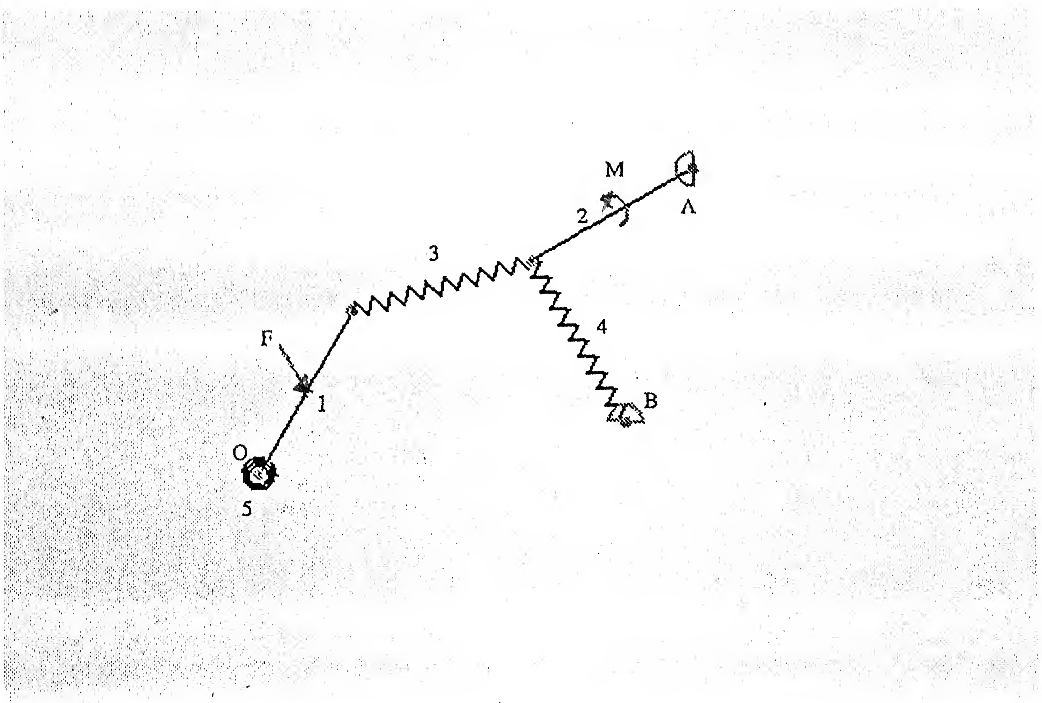
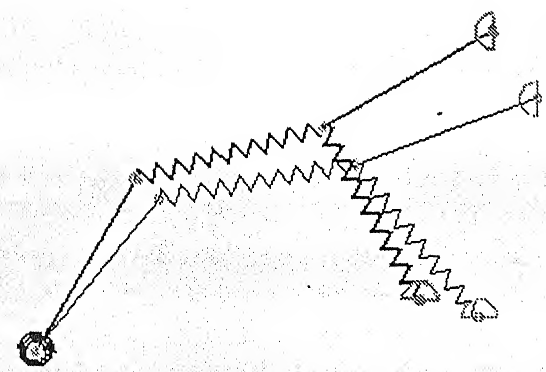
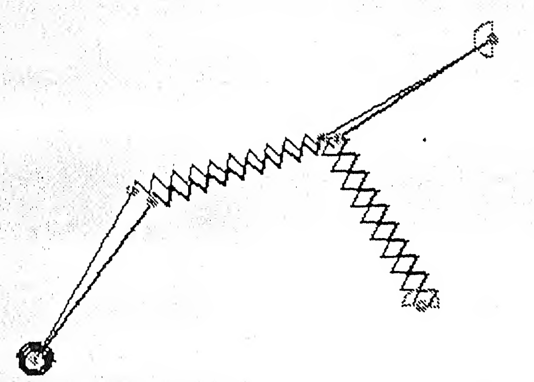


Figure 5.8: Example 4 : Mechanical system



(a) Initial Configuration (Ex. 4)



(b) Equilibrium Configuration

Figure 5.9: Example 4 : Results

Example 5 :

* Mechanical system having 3 rigid and 3 deformable members.
* It is a statically indeterminate system and is shown in Fig. 5.10.
* Link Lengths : $l_1 = 10$, $l_2 = 10$, $l_3 = 10$, PQ = 15, PR = 12
* Angle $RPQ = 60^0$
* Stiffness : $k_2 = k_3 = 60$ N/cm $k_5 = 2000$ N-cm/rad
* Ground connection coordinates : A(25.82, 20.58), B(28.32, -1.43)
* Force : 100 N at 5 cm from A along link 1, Orientation : -60^0
* Moment : 100 N-cm at link 4
* User input : [20, -40, -30, -50]
* Initial correct configuration : [30, -44.99, -40, -30]
* Equilibrium Configuration : [23.1, -46.84, -45.34, -18.98]
* Final spring lengths : $l_3 = 10.56$ and $l_4 = 9.47$ cm
* Torsion spring deformation : 0.12044 rad

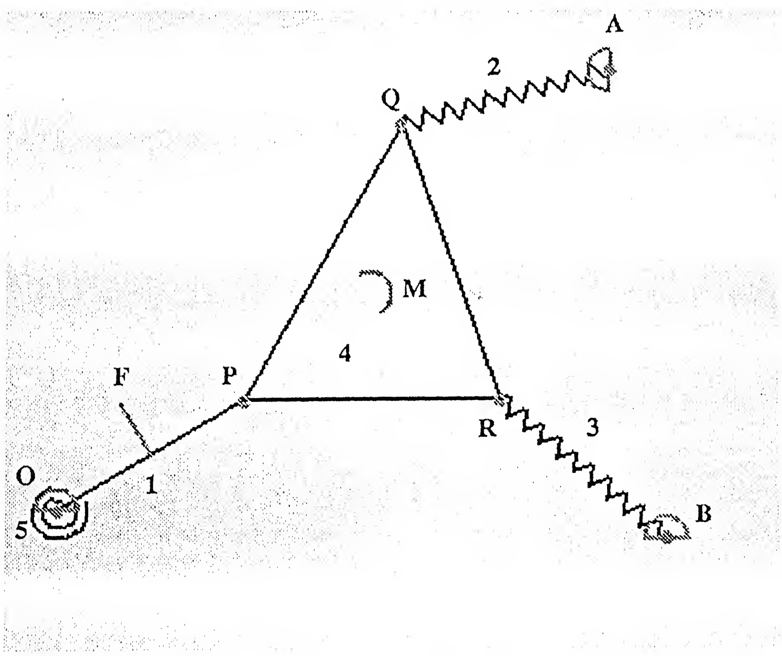
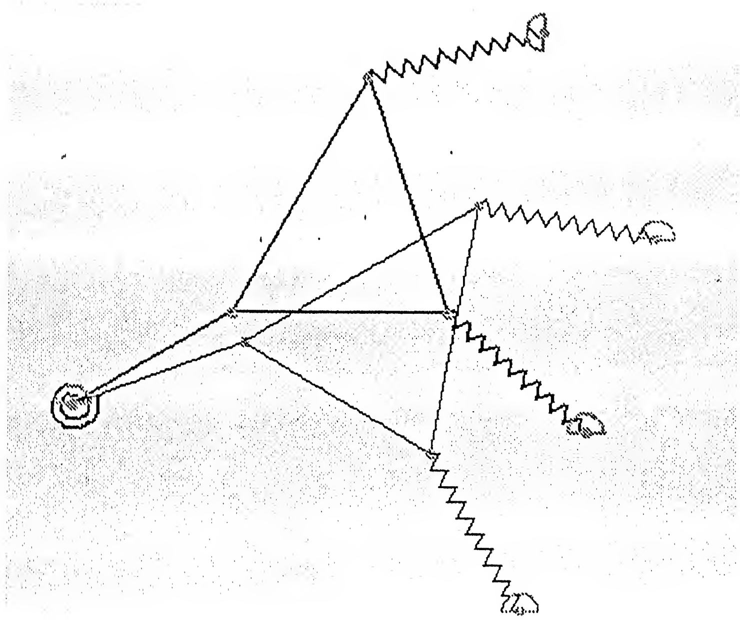
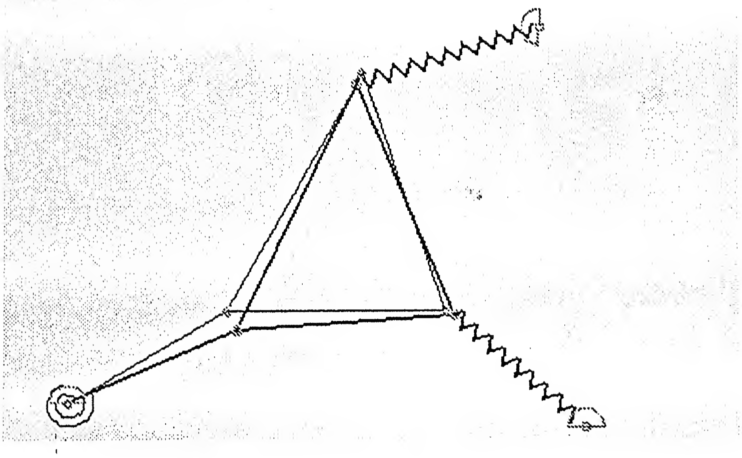


Figure 5.10: Example 5 : Mechanical system



(a) Initial configuration (Ex. 5)



(b) Equilibrium configuration (Ex. 5)

Figure 5.11: Example 5 : Results

Example 6 :

* **Spatial Mechanical system** having 3 rigid and 2 deformable members.
* It is a statically indeterminate system and is shown in Fig. 5.12.
* Link Lengths : $l_1 = 15$, $l_2 = 10$, $l_4 = 12.5$
* Stiffness : $k_3 = 60$ N/cm $\qquad k_4 = 2000$ N-cm/rad
* Ground connection coordinates : A(30.0, 0.0, 6.0)
* Direction cosines for axes of revolute joint :

R_1 (0.1227, 0.1227, 0.9848) and R_2 (-0.2645, 0.2438, 0.933)

* Moment : 90 N-cm at link 1
* User input : [30, -20, -30, 20]
* Initial correct configuration : [32.0637, -28.6109, -56.853, 15.7712]
* Equilibrium Configuration : [34.642, -36.1767, -50.9071, 14.5569]

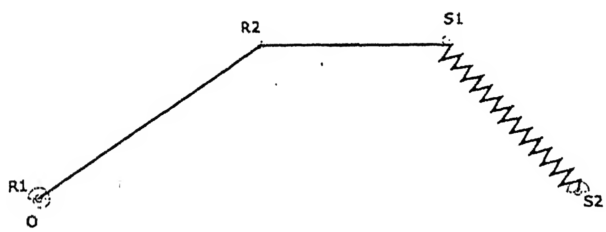
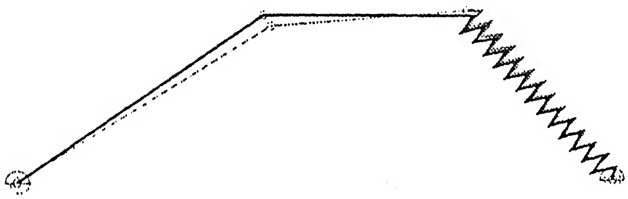


Figure 5.12: **Example 6 : Mechanical system**

(a) Initial configuration



(b) Equilibrium configuration

Figure 5.13: Example 6 : Results

Example 7 :

* **Spatial Mechanical system** having 3 rigid and 3 deformable members.
* It is a statically indeterminate system and is shown in Fig. 5.14.
* Link Lengths : $l_1 = 10$, $l_3 = 15$, $l_4 = 12.5$
* Stiffness : $k_3 = k_4 = 60$ N/cm $\quad k_5 = 1000$ N-cm/rad
* A(15.69, -1.841, 8.604), B(20.248, 1.724, 3.066)
* Direction cosines for axis of revolute joint : (0.1, 0.1, 0.98994)
* Force : (135.95, -63.4, 0) at 5 cm from A along link 1
* Moment : 100 N-cm at link 1
* User input : [25, 5, -45, -35, -40, -25]
* Initial correct configuration : [29.999, 5, -9.999, -40, -44.999, -19.999]
* Equilibrium Configuration : [24.4, 4.4788, -41.14, -39.06, -36.14, -19.79]
* Final spring lengths : $l_3 = 15$, $l_4 = 11.8765$ cm

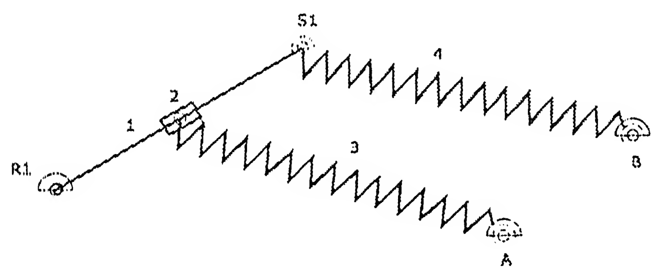
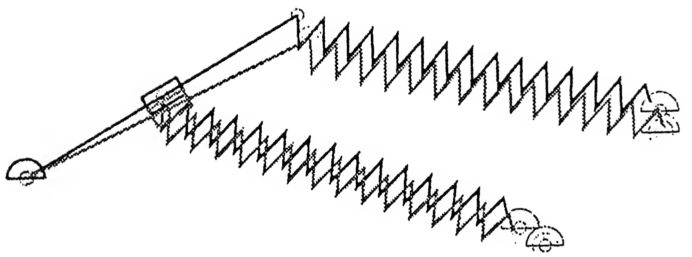
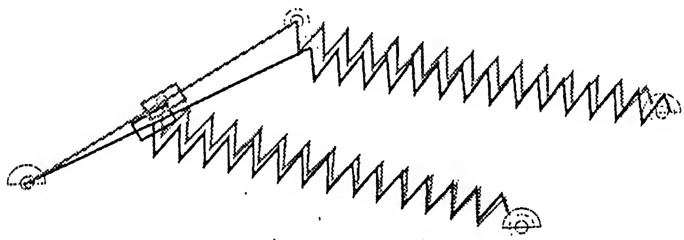


Figure 5.14: **Example 7 : Mechanical system**



(a) Initial configuration



(b) Equilibrium configuration

Figure 5.15: **Example 7 : Results**

For the mechanical system in Example (7), forces of different magnitude is applied. In this case coordinates (X, Y, Z), of endpoint of the link 1, and distance of slider along link 1 from ground (L), defines the configuration of system completely. They are plotted against force magnitude.

Stiffness : $k_3 = k_4 = 60$ N/cm $\qquad\qquad k_5 = 600$ N-cm/rad

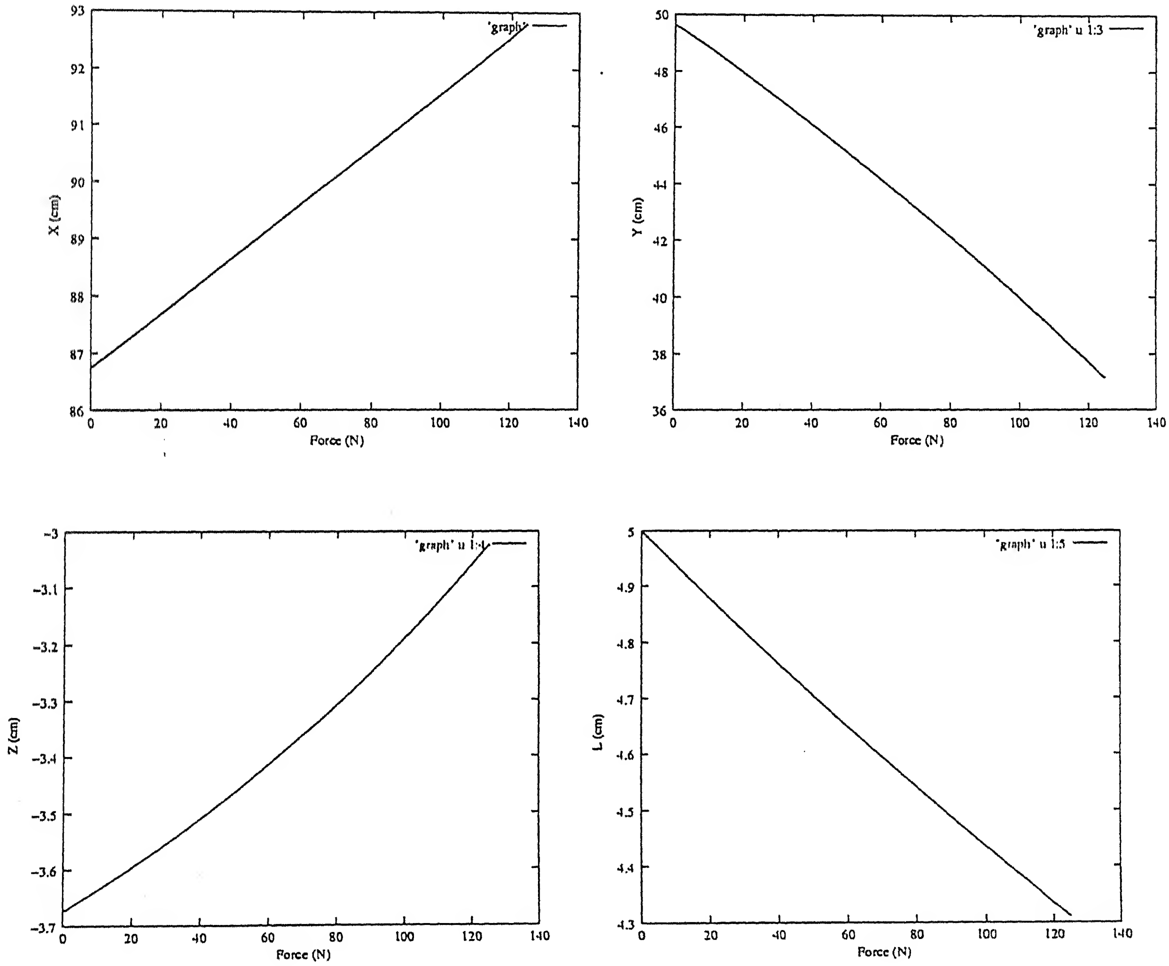


Figure 5.16: **Graphs for Example 7**

Chapter 6

Conclusion

6.1 Summary

In the present work, the static equilibrium analysis of compliant mechanical systems is performed. The systems considered are user defined planar/spatial systems. Both statically determinate as well as statically indeterminate systems are analysed.

Generalised coordinates used are relative coordinates. The system is modeled using loop closure equations and joint transformation matrices. In case of complicated planar and spatial systems, providing accurate values of relative coordinates is a difficult task for the user. Hence, in the first phase of analysis (*assembly*), correct initial configuration of the system is obtained. Matlab routine *fmins* is used for minimisation of objective function.

In static equilibrium analysis, the compliant members are modeled as linear and torsion springs. The external loading is converted into the generalised force and potential energy function of the system is formulated. It is subjected to constraints that the lengths of rigid links remain constant throughout the analysis. This constrained optimisation problem is solved by *constr* function in Matlab optimisation toolbox. Finally, equilibrium configuration in graphical form is displayed. Some results are generated for both planar and spatial cases, with various kinematic joints. The results are found to be satisfactory in the sense that they do give consistent solutions.

Thus, from the results obtained we can conclude that, the methodology suggested for static equilibrium analysis is general and computationally efficient. Main reason for this is the choice of relative coordinates, which leads to lesser number of constraint equations compared to Cartesian coordinates.

6.2 Future Scope

Now, the analysis is fairly general. It can analyse planar/spatial and statically determinate as well as statically indeterminate systems. Hence the next step in this research will be to consider combined loading of members subjected to torsion, moment and shear along with axial load. Detecting singular situations can be another area of research. Also systems analysed in present work are idealised. So, considering friction effects at joints will lead to more practical approach.

Appendix A

The user input for a sample problem is given below. The sample problem is as shown in the figure. The information asked by Software is given in *italics.* Input provided by the user is given in **bold** type. Explanation is given in sans serif type style.

Enter total no. of links including base and torsional springs : **7**

Is it Planer(1) or Spatial(2) System : **1**

Enter total no. of ground connections : **2**

Enter max. no. of links in loop considering all loops : **4**

Enter loop connectivity matrix :

3 1 2 4 1 0

4 1 2 5 3 1

First column contains no. of links in a loop. Subsequent columns contains serially link numbers in that loop followed by joint at the ground connection. If any column is remaining it will contain zero as entry.

Joint terminology for 2D :

Revolute Joint - 1 , Prismatic Joint - 2

Joint terminology for 3D :

Spherical Joint - 1 , Prismatic Joint - 2

Revolute Joint - 3 , Cylindrical Joint - 4 , Screw Joint - 5

Enter link input matrix :

1 10.0 0 30 0

1 15.0 0 -20 0

2 09.0 0 -60 0

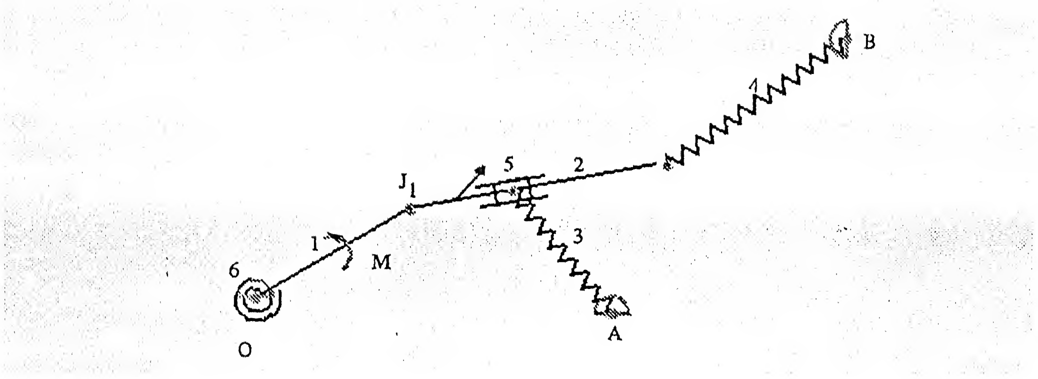
2 12.5 0 25 0

5 06.0 0 0 0

4 00.0 0 0 0

1st Column : Link Type (1 - Straight rigid link, 2 - Linear spring,

3 - Ternary link, 4 - Torsion spring, 5 - Slider)



Mechanical System

2nd Column : Link length

3rd Column : Flag 1 or 2 for ternary link

4th Column : Relative angle with respect to previous link

5th Column : Angle between edges of ternary link

Note : Ternary link is represented by two Straight links with included angle between them as constant.

Enter torsion connectivity : **1 2**

Number of two links which are connected by torsion spring.

Enter Ground connection coordinates :

33.671 14.774

20.354 -0.85251

1st Row : X,Y coordinates of 1st ground connection

2nd Row : X,Y coordinates of 2nd ground connection

Enter no. of forces and no. of moments acting : **1 1**

Enter input force data :

2 0 600 25 4

1st Column : Link no., 2nd Column : 0 if not acting on ternary link

3rd Column : Force Magnitude, 4th Column : Angle w.r.t. positive X axis

5th Column : Distance from previous link

Enter input moment data : **1 -300**

1st Column : Link no., 2nd Column : Moment magnitude

Enter Stiffness of linear springs : **60 60**

Enter Stiffness of torsion spring : **400**

Appendix B

Functions used in the code for Planar Analysis :

- **sp_len :** *Function to calculate linear spring lengths.*
- **constraint :** *It models all the joints. Output of this function is ground connection coordinates.*
- **force_co :** *Coordinates of point of application of force are calculated here.*
- **sp_pe :** *It calculates P.E. due to compliant members.*
- **g_force :** *Here Jacobian, generalised force and P.E. due to it are calculated. Jacobian is modified in each iteration.*
- *All the above functions are called in main program.*
- *Command to run main program :* x = *constr('filename',* v*, 10)*
- v *is initial guess vector of relative coordinates and output vector is* x*.*

Functions used in the code for Spatial Analysis :

- **joint :** *Converts g.c. vector to jointwise g.c.*
- **sp_con :** *Calculates link connectivity in loop and finds loop of which linear spring is a link.*
- **rev_joint :** *Models revolute joint.*
- **sp_len3 :** *Calculates linear sping lengths.*
- **constraint :** *It models all the joints. Output of this function is ground connection coordinates.*

- **co_ord_f :** *Coordinates of point of application of force are calculated here.*

- **sp_pe3 :** *It calculates P.E. due to compliant members.*

- **g_force3 :** *Here Jacobian, generalised force and P.E. due to it are calculated. Jacobian is modified in each iteration.*

- *All the above functions are called in main program.*

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